Natural deduction

In natural deduction, we have such a collection of proof rules. They allow us to infer formulas from other formulas. By applying these rules in succession, we may infer a conclusion from a set of premises. Let's see how this works. Suppose we have a set of formulas4 φ1, φ2, φ3, ... , φn, which we will call premises, and another formula, ψ, which we will call a conclusion. By applying proof rules to the premises, we hope to get some more formulas, and by applying more proof rules to those, to eventually obtain the conclusion. This intention we denote by φ1, φ2,...,φn ψ. This expression is called a sequent; it is valid if a proof for it can be found. The sequent for Examples 1.1 and 1.2 is p ∧ ¬q → r, ¬r, p q. Constructing such a proof is a creative exercise, a bit like programming. It is not necessarily obvious which rules to apply, and in what order, to obtain the desired conclusion. Additionally, our proof rules should be carefully chosen; otherwise, we might be able to 'prove' invalid patterns of argumentation. For

Lower-case

- **φ phi**
- **ψ psi**
- **χ chi**
- **η eta**
- **α alpha**
- **β beta**
- **γ gamma**

Upper-case

- **Φ Phi**
- **Ψ Psi**
- **Γ Gamma**
- **Δ Delta**

example, we expect that we won't be able to show the sequent p, q p ∧ ¬q. For example, if p stands for 'Gold is a metal.' and q for 'Silver is a metal,' then knowing these two facts should not allow us to infer that 'Gold is a metal whereas silver isn't.'

Rules for natural deduction

The rules for conjunction

Our first rule is called the rule for conjunction (∧): and-introduction. It allows us to conclude φ ∧ ψ, given that we have already concluded φ and ψ separately. We write this rule as

$$\frac{\varphi \quad \psi}{\varphi \wedge \psi}\ {\wedge i}$$

Above the line are the two premises of the rule. Below the line goes the conclusion. (It might not yet be the final conclusion of our argument; we might have to apply more rules to get there.) To the right of the line, we write the name of the rule; $\wedge i$ is read ‘and-introduction’. Notice that we have introduced a $\wedge$ (in the conclusion) where there was none before (in the premises). For each of the connectives, there is one or more rules to introduce it and one or more rules to eliminate it. The rules for and-elimination are these two:

$$\frac{\varphi \wedge \psi}{\varphi} \; \wedge e1$$

$$\frac{\varphi \wedge \psi}{\psi} \; \wedge e2$$

The rule $\wedge e1$ says: if you have a proof of $\varphi \wedge \psi$, then by applying this rule you can get a proof of φ. The rule $\wedge e2$ says the same thing, but allows you to conclude ψ instead. Observe the dependences of these rules: in the first rule of (1.1), the conclusion φ has to match the first conjunct of the premise, whereas the exact nature of the second conjunct ψ is irrelevant. In the second rule it is just the other way around: the conclusion ψ has to match the second conjunct ψ and φ can be any formula. It is important to engage in this kind of pattern matching before the application of proof rules.